

THERMODYNAMIC FORMALISM FOR COUNTABLE TO ONE MARKOV SYSTEMS

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ABSTRACT. For countable to one transitive Markov systems we establish thermodynamic formalism for non-Hölder potentials in nonhyperbolic situations. We present a new method for the construction of conformal measures that satisfy the weak Gibbs property for potentials of weak bounded variation and show the existence of equilibrium states equivalent to the weak Gibbs measures. We see that certain periodic orbits cause a phase transition, non-Gibbsianness and force the decay of correlations to be slow. We apply our results to higher-dimensional maps with indifferent periodic points.

§0. INTRODUCTION

Thermodynamic formalism for hyperbolic systems was satisfactorily established with Bowen's program ([2]). The existence of generating finite Markov partitions and analysis of Ruelle-Perron-Frobenius operators associated to Hölder potentials allow one to show the existence of unique equilibrium states that satisfy the Gibbs property (in the sense of Bowen) and exponential decay of correlations. Also, the pressure functions are analytic and there is no possibility of *phase transition* (non-uniqueness of equilibrium states). Furthermore, the analyticity problem is strongly related to multifractal problems and the zero of Bowen's equation determines the Hausdorff dimension of limit sets arising from certain iterated functional systems ([4], [8], [12], [22]). On the other hand, phase transition, failure of the Gibbs property and slow decay of correlations can be observed for many complex systems which exhibit common phenomena in transition to turbulence (the so-called *Intermittency*). In this paper we shall construct mathematical models which exhibit such phenomena and for this purpose we shall establish thermodynamic formalism for non-Hölder potentials in nonhyperbolic situations in the following sense: generating Markov partitions are countable partitions and dynamical instability is *subexponential* (subexponential decay of cylinder sizes). More specifically, for countable to one transitive Markov systems we shall construct conformal measures ν that are *weak Gibbs* measures for potentials ϕ of *weak bounded variation* (WBV) (see definitions in §1) and show the existence of equilibrium states μ for ϕ equivalent to the weak Gibbs measures ν . The conformal measures ν associated to ϕ play important roles as reference measures from the physical point of view, and absolute continuity of equilibrium states allows one to describe statistical properties of observable phenomena in the physical sense. In order to clarify typical reasons for

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phase transition, non-Gibbsianness and slow decay of correlations, we introduce in §4 a notion of *indifferent periodic point* associated to potentials ϕ of WBV. Those periodic points cause failure of summable variation for potentials ϕ and failure of bounded distortion of local Jacobians with respect to the weak Gibbs measure ν for ϕ (Proposition 4). Then a construction of weak Gibbs measures ν for ϕ admitting indifferent periodic points implies “subexponential instability” in terms of cylinder measures (Proposition 3). Furthermore, a construction of equilibrium states μ for such ϕ equivalent to ν (Theorem 6, Lemma 15) allows us to show both phase transition (Corollary 2, Theorem 8) and non-Gibbsianness of equilibrium states (Theorem 5). In particular, our results are applicable to the following piecewise C^1 -smooth countable to one Markov maps T defined on bounded regions $X \subset \mathbb{R}^d$ with indifferent periodic points ($T^q x_0 = x_0, |\det DT^q(x_0)| = 1$) for which the potentials $-\log |\det DT|$ satisfy neither summable variation nor bounded distortion, so that previous results cannot be applicable.

Example A. (Inhomogeneous Diophantine approximations [13], [15], [16], [17], [19], [20], [21]). Let $X = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq 1, -y \leq x < -y + 1\}$ and define $T : X \rightarrow X$ by

$$T(x, y) = \left(\frac{1}{x} - \left\lfloor \frac{1-y}{x} \right\rfloor + \left\lfloor -\frac{y}{x} \right\rfloor, -\left\lfloor -\frac{y}{x} \right\rfloor - \frac{y}{x} \right),$$

where $[x] = \max\{n \in \mathbb{Z} \mid n \leq x\}$ ($x \in \mathbb{N}$) and $[x] = \max\{n \in \mathbb{Z} \mid n < x\}$ ($x \in \mathbb{Z} \setminus \mathbb{N}$). This map admits indifferent periodic points $(1, 0)$ and $(-1, 1)$ with period 2, i.e., $|\det DT^2(1, 0)| = |\det DT^2(-1, 1)| = 1$ and is related to a Diophantine approximation problem of inhomogeneous linear class. \square

Example B (A complex continued fraction [5], [12], [15], [22]). We can define a complex continued fraction transformation $T : X \rightarrow X$ on the diamond-shaped region $X = \{z = x_1\alpha + x_2\bar{\alpha} : -1/2 \leq x_1, x_2 \leq 1/2\}$, where $\alpha = 1 + i$, by $T(z) = 1/z - [1/z]_1$. Here $[z]_1$ denotes $[x_1 + 1/2]\alpha + [x_2 + 1/2]\bar{\alpha}$, where z is written in the form $z = x_1\alpha + x_2\bar{\alpha}$, $[x] = \max\{n \in \mathbb{Z} \mid n \leq x\}$ ($x \in \mathbb{N}$) and $[x] = \max\{n \in \mathbb{Z} \mid n < x\}$ ($x \in \mathbb{Z} \setminus \mathbb{N}$). This transformation has an indifferent periodic orbit $\{1, -1\}$ of period 2 and two indifferent fixed points at i and $-i$. \square

We recall previous works related to thermodynamic formalism for countable to one Markov systems. For countable Markov shifts, O. Sarig proved the existence of conformal measures and equilibrium states associated to *locally* Hölder potentials defined in [10] and D. Fiebig, U. Fiebig and the author proved the existence of equilibrium states for potentials satisfying bounded distortion ($\sup_{n \geq 1} C_n < \infty$ in the definition of WBV) in [7]. Our main Theorems 4-8 do not satisfy these assumptions and Examples A, B show that they cannot be treated by methods in [10] and in [7]. Furthermore, for higher-dimensional systems that are not symbolic systems we may have crucial difficulties in verifying the positive recurrence condition imposed on potentials in both [10] and [7]. The infinite iterated functional systems that Mauldin and Urbanski studied in [8] correspond to the local inverses of piecewise conformal countable Bernoulli systems in our sense, and the method used in [8] severely relies on the Bernoulli property which fails to hold for Example B. Moreover, a Hölder-type condition, imposed on potentials for the existence of conformal measures and for establishing a variational principle, is not satisfied by the important potentials $-\log |\det DT|$ for both Examples A and B.

In order to prove our theorems, we first give in §2 an appropriate definition of topological pressure for countable to one transitive Markov systems with *finite range structure*. Our definition coincides with the standard one by using periodic points under certain conditions (Lemma 6) which can be easily verified for higher-dimensional examples in §8. We also associate the topological pressure to the spectral radius of the Perron-Frobenius-Ruelle operator (Theorem 3). The essential issue for constructing both weak Gibbs measures ν for ϕ and equilibrium states μ for ϕ equivalent to ν is to derive Schweiger's *jump transformations* T^* over full cylinders (see the definition in §1) with respect to which a *local exponential instability* (05) and a *local bounded distortion* (06) for potentials ϕ are satisfied. Then showing the existence of a zero of a generalized Bowen's equation (GBE) for derived potentials ϕ^* associated to T^* (Lemma 7) allows one to show the existence of conformal measures that are weak Gibbs measures for ϕ (Theorem 4). Under a mild condition which cannot be covered by previous works, we show the existence of a zero of (GBE) in §3 by using a product formula of zeta functions (Proposition 1), which shows a nice relation between zeta functions for the original systems and zeta functions for the jump transformations. We also construct σ -finite conformal measures via induced maps T_A over a single full cylinder A (Theorem 7) in §6 by using some idea that appeared in a previous work by M. Denker and the author [5] in which no evidence of the existence of weak Gibbs conformal measures was given. We establish the existence of equilibrium states μ for ϕ of WBV equivalent to the weak Gibbs measure ν for ϕ via a jump transformation (Theorem 6) in §5 and via induced maps (Lemma 15) in §6. Then we can immediately see that the appearance of indifferent periodic orbits associated to ϕ implies a phase transition, i.e., failure of the uniqueness of equilibrium states (Corollary 2, Theorem 8). We should remark that our construction via induced maps shows the existence of (countably many) mutually singular equilibrium states. In §8 we apply our results to higher-dimensional piecewise C^1 Markov maps with indifferent periodic points. All proofs of results in §§2-3 are postponed to the Appendix.

§1. PRELIMINARIES

Let (X, d) be a compact metric space and let $T : X \rightarrow X$ be a noninvertible map that is not necessarily continuous. Suppose that there exists a countable disjoint partition $Q = \{X_i\}_{i \in I}$ of X such that $\bigcup_{i \in I} \text{int} X_i$ is dense in X and the following properties are satisfied.

- (01) For each $i \in I$ with $\text{int} X_i \neq \emptyset$, $T|_{\text{int} X_i} : \text{int} X_i \rightarrow T(\text{int} X_i)$ is a homeomorphism and $(T|_{\text{int} X_i})^{-1}$ extends to a homeomorphism v_i on $cl(T(\text{int} X_i))$.
- (02) $T(\bigcup_{\text{int} X_i = \emptyset} X_i) \subset \bigcup_{\text{int} X_i = \emptyset} X_i$.
- (03) $\{X_i\}_{i \in I}$ generates \mathcal{F} , the σ -algebra of Borel subsets of X .

We say that the triple $(T, X, Q = \{X_i\}_{i \in I})$ is a piecewise C^0 -invertible system. By (01), $T|_{\text{int} X_i}$ extends to a homeomorphism $(v_i)^{-1}$ on $cl(\text{int} X_i)$ for $i \in I$ with $\text{int} X_i \neq \emptyset$. For notational convenience we denote $(v_i)^{-1} = T|_{cl(\text{int} X_i)}$. Let $\underline{i} = (i_1 \dots i_n) \in I^n$ satisfy $\text{int}(X_{i_1} \cap T^{-1} X_{i_2} \cap \dots \cap T^{-(n-1)} X_{i_n}) \neq \emptyset$. Then we define $X_{\underline{i}} := X_{i_1} \cap T^{-1} X_{i_2} \cap \dots \cap T^{-(n-1)} X_{i_n}$, which is called a cylinder of rank n and write $|\underline{i}| = n$. By (01), $T^n|_{\text{int} X_{i_1 \dots i_n}} : \text{int} X_{i_1 \dots i_n} \rightarrow T^n(\text{int}(X_{i_1 \dots i_n}))$ is a homeomorphism and $(T^n|_{\text{int} X_{i_1 \dots i_n}})^{-1}$ extends to a homeomorphism $v_{i_1} \circ v_{i_2} \circ \dots \circ v_{i_n} = v_{i_1 \dots i_n} : cl(T^n(\text{int} X_{\underline{i}})) \rightarrow cl(\text{int} X_{\underline{i}})$ and $(v_{i_1 \dots i_n})^{-1} = T^n|_{cl(\text{int} X_{i_1 \dots i_n})}$. We impose

on (T, X, Q) the next condition, which gives a nice countable states symbolic dynamics similar to sofic shifts (cf. [15], [16], [17], [19]):

(*Finite Range Structure*). $\mathcal{U} = \{int(T^n X_{i_1 \dots i_n}) : \forall X_{i_1 \dots i_n}, \forall n > 0\}$ consists of finitely many open subsets U_1, \dots, U_N of X . \square

In particular, if (T, X, Q) satisfies the Markov property (i.e., $intX_i \cap intTX_j \neq \emptyset$ implies $intTX_j \supset intX_i$), then $\mathcal{U} = \{int(TX_i) : \forall i \in I\}$ and we say that (T, X, Q) is an *FRS Markov system*. If $X_i \in Q$ satisfies $cl(T(intX_i)) = X$, then X_i is called a *full cylinder*. If all cylinders are full cylinders so that $\mathcal{U} = \{intX\}$, then (T, X, Q) is called a *Bernoulli system*. We assume further the next transitive condition:

(*Transitivity*). $intX = \bigcup_{k=1}^N U_k$ and $\forall l \in \{1, 2, \dots, N\}, \exists 0 < s_l < \infty$ such that for each $k \in \{1, 2, \dots, N\}$, U_k contains an interior of a cylinder $X^{(k,l)}(s_l)$ of rank s_l such that $T^{s_l}(intX^{(k,l)}(s_l)) = U_l$. \square

The transitivity condition allows one to establish the next fact.

Lemma 1. *There exists $0 < S < \infty$ such that $T^S(\bigcup_{l=1}^N intX^{(k,l)}(s_l)) = intX$ and $\forall X_i \in Q, T^{S+1}(intX_i) = intX$.*

Proof of Lemma 1. Since each U_k contains $\bigcup_{l=1}^N intX^{(k,l)}(s_l)$, choosing $S = \prod_{l=1}^N s_l$ is enough to establish the desired fact. \square

Remark (A). If (T, X, Q) is a Markov system, then the transitivity condition implies *aperiodicity* in the following sense: $\exists S > 0$ such that $\forall U_k, U_l \in \mathcal{U}, \forall n > S, \exists \underline{i} = (i_1 \dots i_n)$ with $intX_{\underline{i}} \neq \emptyset$ satisfying $intX_{i_1} \subset U_k$ and $T(int(X_{i_n})) = U_l$.

Definition. We say that ϕ is a potential of *weak bounded variation* (WBV) if there exists a sequence of positive numbers $\{C_n\}$ satisfying $\lim_{n \rightarrow \infty} (1/n) \log C_n = 0$ and $\forall n \geq 1, \forall X_{i_1 \dots i_n} \in \bigvee_{j=0}^{n-1} T^{-j}Q$,

$$\frac{\sup_{x \in X_{i_1 \dots i_n}} \exp(\sum_{j=0}^{n-1} \phi(T^j x))}{\inf_{x \in X_{i_1 \dots i_n}} \exp(\sum_{j=0}^{n-1} \phi(T^j x))} \leq C_n.$$

Define

$$Var_n(T, \phi) := \sup_{Y \in \bigvee_{j=0}^{n-1} T^{-j}(Q)} \sup_{x, y \in Y} |\phi(x) - \phi(y)|.$$

Remark (B). If $Var_n(T, \phi) \rightarrow 0$ as $n \rightarrow \infty$, which implies continuity of ϕ in symbolic distance, then ϕ satisfies the WBV property. Hence if (T, X, Q) is a subshift of finite type, then any continuous functions satisfy the WBV property and if (T, X, Q) is a countable Markov shift, then any uniformly continuous functions ϕ with $Var_1(T, \phi) < \infty$ satisfy the WBV property ([7]).

Let \mathcal{F} be the σ -algebra of Borel sets of the compact space X .

Definition ([17], [18], [20]). A probability measure ν on (X, \mathcal{F}) is called a *weak Gibbs measure* for a function ϕ with a constant P if there exists a sequence $\{K_n\}_{n>0}$ of positive numbers with $\lim_{n \rightarrow \infty} (1/n) \log K_n = 0$ such that ν -a.e. x ,

$$K_n^{-1} \leq \frac{\nu(X_{i_1 \dots i_n}(x))}{\exp(\sum_{j=0}^{n-1} \phi T^j(x) + nP)} \leq K_n,$$

where $X_{i_1 \dots i_n}(x)$ denotes the cylinder containing x .

For a function $\phi : X \rightarrow \mathbb{R}$, we define an operator \mathcal{L}_ϕ by

$$\mathcal{L}_\phi g(x) = \sum_{i \in I} \exp \phi(v_i(x)) g(v_i(x)) 1_{cl(T(int X_i))}(x) \quad (\forall g \in C(X), \forall x \in X).$$

If ϕ satisfies $Var_n(\phi) \rightarrow 0$ ($n \rightarrow \infty$), $\|\mathcal{L}_\phi 1\| := \sup_{x \in X} \mathcal{L}_\phi 1(x) < \infty$ and

(04) $\{v_i\}_{i \in I}$ is an equi-continuous family of partially defined uniformly continuous maps,

then \mathcal{L}_ϕ preserves $C(X)$ (i.e., $\mathcal{L}_\phi : C(X) \rightarrow C(X)$) and is called the *Ruelle-Perron-Frobenius operator*. We remark that (04) is valid if $\sigma(n) = \sup\{\text{diam } Y \mid Y \in \bigvee_{j=0}^{n-1} T^{-j}(Q)\} \rightarrow 0$ as $n \rightarrow \infty$.

We recall the next result, which follows from Theorem 5.1 in [17] and Proposition 2.2 in [18].

Lemma 2 ([17], [18]). *Let (T, X, Q) be a transitive FRS Markov system satisfying $int X \in \mathcal{U}$, and let ϕ be a potential of WBV. Assume that there exist $p > 0$ and a Borel probability measure ν on (X, \mathcal{F}) satisfying $\mathcal{L}_\phi^* \nu = p\nu$, where \mathcal{L}_ϕ^* is the dual of \mathcal{L}_ϕ . Then ν is a weak Gibbs measure for ϕ with $-\log p$.*

Definition. We say that a Borel probability measure ν on X is an *f-conformal measure* if $\frac{d(\nu T)|_{X_i}}{d\nu|_{X_i}} = f|_{X_i}$ ($\forall i \in I$) and $\nu(\bigcup_{i \in I} \partial X_i) = 0$.

In order to show the weak Gibbs property of ν , we use the following formula of the local Jacobians with respect to ν :

$$\frac{d(\nu T)|_{X_i}}{d\nu|_{X_i}} = \exp[\log p - \phi]|_{X_i} \quad (\forall i \in I).$$

Thus for the existence of weak Gibbs measures, it is enough to show the existence of conformal measures (see §3).

Lemma 3 (Theorem 2.2 in [18]). *Let ν be a weak Gibbs measure for ϕ with $-P$. If there exists a T -invariant ergodic probability measure μ equivalent to ν with $H_\mu(Q) < \infty$ and $\phi \in L^1(\mu)$, then $P = h_\mu(T) + \int_X \phi d\mu$.*

In particular, if the constant P is the measure-theoretical pressure, then the existence of a T -invariant ergodic probability measure μ equivalent to the weak Gibbs measure ν for ϕ with $-P$ implies the existence of an equilibrium state for ϕ (see §4). In order to achieve both constructions of conformal measures and equilibrium states, we need to introduce new derived systems which are called *jump transformations* ([13]). Let $B_1 \subset X$ be a union of cylinders of rank 1 of which index i belongs to a subset J of I , and let $D_1 := B_1^c$. Define a function $R : X \rightarrow \mathbb{N} \cup \{\infty\}$ by $R(x) = \inf\{n \geq 0 : T^n x \in B_1\} + 1$ and for each $n > 1$, define inductively

$$B_n := \{x \in X \mid R(x) = n\}, \quad D_n := \{x \in X \mid R(x) > n\} = \bigcap_{m=0}^{n-1} T^{-m} B_1^c.$$

Now we define Schweiger's jump transformation ([13]) $T^* : \bigcup_{n=1}^\infty B_n \rightarrow X$ by $T^* x = T^{R(x)} x$. We denote $X^* := X \setminus (\bigcup_{m=0}^\infty T^{*-m}(\bigcap_{n \geq 0} \{R(x) > n\}))$ and

$$I^* := \bigcup_{n \geq 1} \{(i_1 \dots i_n) \in I^n : X_{i_1 \dots i_n} \subseteq B_n\}.$$

Then it is easy to see that $(T^*, X^*, Q^* = \{X_i\}_{i \in I^*})$ is an FRS Markov system. For a given $\phi : X \rightarrow \mathbb{R}$, we define $\phi^* : \bigcup_{n=1}^\infty B_n \rightarrow \mathbb{R}$ by $\phi^*(x) = \sum_{h=0}^{R(x)-1} \phi T^h(x)$.

Definition. We say that an FRS Markov system satisfies *local exponential instability with respect to B_1* if (05) : $\exists 0 < \gamma^* < 1, \exists 0 < \Gamma^* < \infty$ such that $\forall n \geq 1$,

$$\sigma_{X^*, T^*}(n) = \sup\{\text{diam } Y \mid Y \in \bigvee_{j=0}^{n-1} T^{*-j}(Q^*)\} \leq \Gamma^* \gamma^{*n}.$$

Definition. We say that a potential $\phi : X \rightarrow \mathbb{R}$ satisfies *local bounded distortion with respect to B_1* if there exists $\theta > 0$ such that (06) : $\forall \underline{i} = (i_1 \dots i_{|\underline{i}|}) \in I^*, \exists 0 < L_\phi(\underline{i}) < \infty$ satisfying

$$|\sigma(v_{\underline{i}}(x)) - \sigma(v_{\underline{i}}(y))| \leq L_\phi(\underline{i}) d(x, y)^\theta \quad (\forall x, y \in T^{|\underline{i}|} X_{\underline{i}})$$

and

$$\sup_{\underline{i} \in I^*} \sum_{j=0}^{|\underline{i}|-1} L_\phi(i_{j+1} \dots i_{|\underline{i}|}) < \infty.$$

Under the conditions (05-06), we can easily verify that $\{\phi^* v_{\underline{i}} : \underline{i} \in I^*\}$ is an equi-Hölder continuous family (cf. [19], [20]) and $\sum_{n=1}^\infty \text{Var}_n(T^*, \phi^*) < \infty$. Both conditions (05-06) can be easily verified for all higher-dimensional examples in §8.

In the rest of this section we shall state relations between jump transformations associated to B_1 and induced maps over B_1 . Let $R_{B_1} : B_1 \rightarrow \mathbb{R} \cup \{\infty\}$ be the first return function defined by $R_{B_1}(x) = \inf\{n \geq 1 : T^n x \in B_1\}$. Then we define the induced map T_{B_1} over $\{x \in B_1 : R_{B_1}(x) < \infty\}$ by $T_{B_1} x = T^{R_{B_1}(x)} x$ and the induced potential $\phi_{B_1} : \{x \in B_1 : R_{B_1}(x) < \infty\} \rightarrow \mathbb{R}$ by $\phi_{B_1}(x) = \sum_{h=0}^{R_{B_1}(x)-1} \phi T^h(x)$. Then we can immediately see the following facts.

Lemma 4. (4-1) $R_{B_1} = R \circ T|_{B_1}$,

$$(4-2) \quad \phi_{B_1} - sR_{B_1} = (\phi^* - sR) \circ T|_{B_1} + (\phi - \phi \circ T_{B_1}),$$

$$(4-3) \quad \sum_{m=0}^{n-1} (\phi_{B_1} - sR_{B_1}) \circ T_{B_1}^m = \sum_{m=0}^{n-1} (\phi^* - sR) \circ T^{*m} \circ T|_{B_1^*} + (\phi - \phi \circ T_{B_1}^n),$$

where $B_1^* := B_1 \setminus (\bigcup_{m=0}^\infty T_{B_1}^{-m}(\{R_{B_1}(x) = \infty\}))$.

Lemma 5 (Lemma 4.1 in [18]). *Suppose that B_1 consists of full cylinders. Then for any T -invariant probability measure m with $m(B_1) > 0$, $m_{B_1} := \frac{m|_{B_1}}{m(B_1)}$ is a T_{B_1} -invariant probability measure and $m^* := m_{B_1} T|_{B_1}^{-1}$ is a T^* -invariant probability measure. m can be written in terms of m^* by Schweiger's formula (see (3) in §5) and in terms of m_{B_1} by Kac's formula (see Lemma 16).*

§2. TOPOLOGICAL PRESSURE FOR POTENTIALS OF WEAK BOUNDED VARIATION

Let (T, X, Q) be a transitive FRS Markov system and let $\phi : X \rightarrow \mathbb{R}$ be a potential of WBV. For each $n > 0$ and for each $U \in \mathcal{U}$ we define the following partition functions :

$$Z_n(U, \phi) := \sum_{\underline{i}: |\underline{i}|=n, \text{int}(TX_{i_n})=U \supset \text{int} X_{i_1}} \sum_{v_{\underline{i}} x = x \in \text{cl}(\text{int} X_{\underline{i}})} \exp\left[\sum_{h=0}^{n-1} \phi T^h(x)\right],$$

$$\overline{Z}_n(U, \phi) = \sum_{\underline{i}: |\underline{i}|=n, \text{int}(TX_{i_n})=U \supset \text{int} X_{i_1}} \sup_{x \in X_{\underline{i}}} \exp\left[\sum_{h=0}^{n-1} \phi T^h(x)\right]$$

and

$$\underline{Z}_n(U, \phi) = \sum_{\underline{i}: |\underline{i}|=n, \text{int}(TX_{i_n})=U \supset \text{int}X_{i_1}} \inf_{x \in X_{\underline{i}}} \exp\left[\sum_{h=0}^{n-1} \phi T^h(x)\right].$$

We further define

$$Z_n(\phi) := \sum_{\underline{i}: |\underline{i}|=n, \text{int}(TX_{i_n}) \supset \text{int}X_{i_1}} \sum_{v_{\underline{i}}x=x \in \text{cl}(\text{int}X_{\underline{i}})} \exp\left[\sum_{h=0}^{n-1} \phi T^h(x)\right].$$

We shall define the topological pressure as the asymptotic growth rates of these partition functions.

Theorem 1 (Topological pressure for potentials of WBV). *Let (T, X, Q) be a transitive FRS Markov system and let ϕ be a potential of WBV. For each $U \in \mathcal{U}$, $\lim_{n \rightarrow \infty} \frac{1}{n} \log \bar{Z}_n(U, \phi)$, $\lim_{n \rightarrow \infty} \frac{1}{n} \log \underline{Z}_n(U, \phi)$, $\lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n(U, \phi)$ exist and do not depend on U . Furthermore, the limits coincide with $\lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n(\phi)$.*

We call the limit $P_{\text{top}}(T, \phi) := \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n(\phi)$ the *topological pressure* for ϕ . The next fact can be verified easily.

Lemma 6. *Under the next condition, $Z_n(\phi)$ coincides with the usual partition function defined by $\sum_{T^n x=x, x \in X} \exp[\sum_{h=0}^{n-1} \phi T^h(x)]$.*

- (1) *For $x_0 \in X_{i_1 \dots i_n}$ with $T^n x_0 = x_0$, either $x_0 \in \text{int}X_{i_1 \dots i_n}$ or $x_0 \notin \text{cl}X_{j_1 \dots j_n}$ for $(j_1 \dots j_n) \neq (i_1 \dots i_n)$.*

Let \mathcal{V} be the finite disjoint partition generated by \mathcal{U} . We should claim that if a periodic point x_0 with period n is contained in a cylinder $X_{i_1 \dots i_n}$ satisfying $X_{i_1 \dots i_n} \subset \text{int}V$ for some $V \in \mathcal{V}$, then $x_0 \notin \partial X_{i_1 \dots i_n}$. If not, we have a contradiction to $x_0 \in \text{int}V$ because of $x_0 \in T^n(\partial X_{i_1 \dots i_n}) = \partial(T^n X_{i_1 \dots i_n})$. By using this fact, we will see that all higher-dimensional examples in §8 satisfy (1). The Artin-Mazur-Ruelle zeta function $\zeta_{T, \phi}(z)$ is defined by $\zeta_{T, \phi}(z) = \exp[\sum_{n=1}^{\infty} \frac{z^n}{n} Z_n(\phi)]$. Then the radius of convergence of $\zeta_{T, \phi}(z)$ is given by $\rho_{\phi} = \exp[\limsup_{n \rightarrow \infty} \frac{1}{n} \log Z_n(\phi)]^{-1}$. We define

$$\mathcal{W}(T) := \{\phi : X \rightarrow \mathbb{R} \mid \phi \text{ satisfies WBV and } P_{\text{top}}(T, \phi) < \infty\}$$

and

$$\mathcal{W}_B(T) := \{\phi \in \mathcal{W}(T) \mid \text{Var}_n \phi \rightarrow 0 \ (n \rightarrow \infty), \|\phi\| := \sup_{x \in X} |\phi(x)| < \infty\}.$$

We can easily see that the pressure function $P_{\text{top}}(T, \cdot) : \mathcal{W}(T) \rightarrow \mathbb{R}$ satisfies continuity, convexity and $\forall \phi_1, \phi_2 \in \mathcal{W}(T)$, $P_{\text{top}}(T, \phi_1 + \phi_2) \leq P_{\text{top}}(T, \phi_1) + P_{\text{top}}(T, \phi_2)$. Furthermore, by applying Theorem 2.4 in [7] we have the following fact.

Theorem 2. $\mathcal{W}_B(T)$ is a Banach space and $P_{\text{top}}(T, \cdot) : \mathcal{W}_B(T) \rightarrow \mathbb{R}$ is a Lipschitz continuous convex function.

Definition. If an FRS Markov system (T, X, Q) satisfies that $\forall U \in \mathcal{U}, \exists X_i \in Q$ such that $X_i \subset U$ and $T(\text{int}X_i) = \text{int}X$, then (T, X, Q) is called a *strongly transitive* FRS Markov system.

Theorem 3 (Topological pressure and the spectral radius). *Let (T, X, Q) be a strongly transitive FRS Markov system satisfying $\mathcal{U} \cap \mathcal{V} \neq \emptyset$. Let ϕ be a potential of weak bounded variation. Then $\forall U \in \mathcal{U} \cap \mathcal{V}$ and $\forall x \in U$, $\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{L}_{\phi}^n 1_U(x) = P_{\text{top}}(T, \phi)$. Furthermore, $\lim_{n \rightarrow \infty} \frac{1}{n} \log \|\mathcal{L}_{\phi}^n 1\| = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|\mathcal{L}_{\phi}^n\| = P_{\text{top}}(T, \phi)$.*

We can easily verify all conditions in Theorem 3 for examples in §8.

§3. THE CONSTRUCTION OF WEAK GIBBS CONFORMAL MEASURES

Let (T, X, Q) be a transitive FRS Markov system and let $\phi \in \mathcal{W}(T)$. Suppose that there exists a union of full cylinders $B_1(\subset X)$ with respect to which (T, X, Q) satisfies local exponential instability and ϕ satisfies local bounded distortion. For the derived potential $\phi^*(x) = \sum_{h=0}^{R(x)-1} \phi T^h(x)$ we define

$$Z_n(\phi^*) := \sum_{(\underline{i}_1 \dots \underline{i}_n) \in I^{*n} : \text{int}(T^* X_{\underline{i}_n}) \supset \text{int} X_{\underline{i}_1} \dots \underline{i}_n} \sum_{v_{\underline{i}_1} \dots \underline{i}_n, x \in \text{cl}(\text{int} X_{\underline{i}_1} \dots \underline{i}_n)} \exp \left[\sum_{m=0}^{n-1} \phi T^{*m}(x) \right].$$

Then by Theorem 1, summable variations of ϕ^* allow one to show that

$$\exists \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n(\phi^*) := P_{\text{top}}(T^*, \phi^*) \in (-\infty, \infty].$$

Theorem 4 (A construction of conformal measures via jump transformations). *Let (T, X, Q) be a transitive FRS Markov system and let $\phi \in \mathcal{W}(T)$. Suppose that there exists a union of full cylinders $B_1(\subset X)$ with respect to which (T, X, Q) satisfies local exponential instability and ϕ satisfies local bounded distortion. Assume further that $\|\mathcal{L}_{\phi^*} - R \min\{0, P_{\text{top}}(T^*, \phi^*)\} 1\| < \infty$. Then there exists a Borel probability measure ν on X supported on X^* satisfying*

$$\frac{d\nu T}{d\nu}|_{X_i} = \exp[P_{\text{top}}(T, \phi) - \phi](\forall i \in I)$$

and $\nu(\bigcup_{i \in I} \partial X_i) = 0$.

As we have announced in §0, for constructing a weak Gibbs measure for ϕ of WBV, we shall consider the following generalized Bowen's equation:

$$(\text{GBE}) : P_{\text{top}}(T^*, \phi^* - sR) = 0.$$

The existence of a zero of the equation (GBE) follows from the standard argument in the case when $0 \leq P_{\text{top}}(T^*, \phi^*) < \infty$ because of continuity of the function $s \rightarrow P_{\text{top}}(T^*, \phi^* - sR)$ on $\text{int}\{s \in \mathbb{R} \mid P_{\text{top}}(T^*, \phi^* - sR) \in \mathbb{R}\}$ (see Lemma 14). We should notice that the uniqueness of the zero of (GBE) follows from the “strictly” decreasing property of the function, $s \rightarrow P_{\text{top}}(T^*, \phi^* - sR)$ in the standard situation. Here we have no evidence of it although the function is decreasing. If $P_{\text{top}}(T^*, \phi^*) < 0$, then under the assumption $P_{\text{top}}(T^*, \phi^* - RP_{\text{top}}(T^*, \phi^*)) < \infty$ we see that

$$P_{\text{top}}(T^*, \phi^* - RP_{\text{top}}(T^*, \phi^*)) \geq P_{\text{top}}(T^*, \phi^* - P_{\text{top}}(T^*, \phi^*)) = 0$$

and so we can reduce to the previous case. If $P_{\text{top}}(T^*, \phi^*) = \infty$, then we cannot use the standard argument. Now we come to state the next key lemma, which allows one to establish Theorem 4.

Lemma 7 (The existence of a zero of (GBE)). (i) *If $0 \leq P_{\text{top}}(T^*, \phi^*) < \infty$, then $P_{\text{top}}(T, \phi) \geq 0$ and $\exists s_0 \geq 0$ satisfying $P_{\text{top}}(T^*, \phi^* - s_0 R) = 0$.*

(ii) *If $P_{\text{top}}(T^*, \phi^*) < 0$ and $P_{\text{top}}(T^*, \phi^* - RP_{\text{top}}(T^*, \phi^*)) < \infty$, then $P_{\text{top}}(T, \phi - P_{\text{top}}(T^*, \phi^*)) \geq 0$ and $\exists s_0 \geq 0$ satisfying $P_{\text{top}}(T^*, (\phi - P_{\text{top}}(T^*, \phi^*))^* - s_0 R) = 0$.*

(iii) *If $\sup\{s \in \mathbb{R} : P_{\text{top}}(T^*, \phi^* - sR) = \infty\} = \min\{0, P_{\text{top}}(T^*, \phi^*)\}$, then $\exists s_0 \geq \min\{0, P_{\text{top}}(T^*, \phi^*)\}$ such that $P_{\text{top}}(T^*, \phi^* - s_0 R) = 0$.*

We recall the formal power series $\zeta_{T,\phi}(z) = \exp[\sum_{n=1}^{\infty} \frac{z^n}{n} Z_n(\phi)]$, which is called the Artin-Mazur-Ruelle zeta function. The next product formula of zeta functions plays an important role in proving Lemma 7.

Proposition 1 (cf. [17]). *We can write*

$$(2) \quad \zeta_{T,\phi}(\exp(-s)) = \zeta_{T^*,\phi^*-sR}(1) \times \zeta_{T|_{\cap_{n>0} D_n},\phi}(\exp(-s)).$$

Corollary 1. *If $s > P_{\text{top}}(T, \phi)$, then $P_{\text{top}}(T^*, \phi^* - sR) \leq 0$.*

By Theorem 3, the assumption $\|\mathcal{L}_{\phi^*-R\min\{0, P_{\text{top}}(T^*, \phi^*)\}} 1\| < \infty$ implies either $0 \leq P_{\text{top}}(T^*, \phi^*) < \infty$ or $P_{\text{top}}(T^*, \phi^* - RP_{\text{top}}(T^*, \phi^*)) < \infty$ is satisfied. Hence it follows from Lemma 7 that $\exists s_0 \geq \min\{0, P_{\text{top}}(T^*, \phi^*)\}$ satisfying $P_{\text{top}}(T^*, \phi^* - s_0R) = 0$ and $\|\mathcal{L}_{\phi^*-s_0R} 1\| < \infty$. Since $Q^* = \{X_i\}_{i \in I^*}$ consists of full cylinders and summability of variations $\sum_{n=1}^{\infty} \text{Var}_n(T^*, \phi^* - s_0R) < \infty$ is valid, we can apply P. Walter's argument in [14] to show the existence of an $\exp[s_0R - \phi^*]$ -conformal measure with respect to T^* .

Lemma 8. *There exists a Borel probability measure ν on X satisfying $\mathcal{L}_{\phi^*-s_0R}^* \nu = \nu$ and $\nu(\text{int} X^*) = 1$.*

In §9 we shall show the existence of an $\exp[s_0 - \phi]$ -conformal measure for the zero s_0 of (GBE) by using the conformal measure ν on X^* and show $s_0 = P_{\text{top}}(T, \phi)$, which implies uniqueness of the zero of (GBE). At the end of this section, we shall consider the case when $\|\mathcal{L}_{\phi^*-R\min\{0, P_{\text{top}}(T^*, \phi^*)\}} 1\| = \infty$. By Theorem 3, if $P_{\text{top}}(T^*, \phi^* - s_0R) = 0$, then there exists sufficiently large n such that $\|\mathcal{L}_{(\phi-s_0)^*}^n 1\| = \|\mathcal{L}_{(\phi-s_0)_n^*} 1\| < \infty$, where

$$(\phi - s_0)_n^* := \sum_{m=0}^{n-1} (\phi - s_0)^* T^{*m} = \sum_{m=0}^{n-1} (\phi^* - s_0R) T^{*m}.$$

We shall introduce a new stopping time (depending on $n \geq 1$) defined on X^* by

$$R_n(x) := \inf\{k \geq n \mid X_{i_1, \dots, i_k}(x) \in \bigvee_{m=0}^{n-1} T^{*-m} Q^*\} = \sum_{m=0}^{n-1} R(T^{*m}(x)).$$

Then a new jump transformation S^* defined by $S^*(x) := T^{R_n(x)}(x)$ is equal to T^{*n} and the next facts can be verified easily.

Lemma 9. (9-1) $R_n(T(x)) = R_n(x) - 1$.

(9-2) $\{x \in X^* \mid R_n(x) = k\} = \{x \in X^* \mid \sum_{m=1}^n |i_m| = k \text{ where } x \in X_{i_1, \dots, i_n} \in \bigvee_{m=0}^{n-1} T^{*-m} Q^*\}$.

(9-3) $(\phi - s_0)_n^*(x) = \phi_n^*(x) - s_0 R_n(x) = \sum_{h=0}^{R_n(x)-1} (\phi - s_0) T^h(x)$.

Now we shall consider a two-parameter family of functions $\{(\phi - s)_n^* \mid (s, n) \in \mathbb{R} \times \mathbb{N}\}$ and the equations $P_{\text{top}}(T^{*n}, (\phi - s)_n^*) = 0$. Applying Theorem 4 gives the next result.

Proposition 2. *Suppose that all conditions in Theorem 3 are satisfied. If there exist $s_0 \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, $P_{\text{top}}(T^{*n}, (\phi - s_0)_n^*) = 0$, then there exists a Borel probability measure ν on X supported on X^* satisfying $\frac{d\nu T}{d\nu}|_{X_i} = \exp[s_0 - \phi](\forall i \in I)$ and $\nu(\bigcup_{i \in I} \partial X_i) = 0$. Furthermore, $s_0 = P_{\text{top}}(T, \phi)$.*

§4. INDIFFERENT PERIODIC POINTS AND NON-GIBBSIANNES

Let (T, X, Q) be a transitive FRS Markov system and let $\phi \in \mathcal{W}(T)$. The next lemma follows from the definition of $P_{\text{top}}(T, \phi)$ directly.

Lemma 10. $P_{\text{top}}(T, \phi) \geq \frac{1}{q} \sum_{h=0}^{q-1} \phi T^h(x_0) (\forall x_0 \in X, T^q x_0 = x_0)$.

Definition. x_0 is called a *generalized indifferent periodic point* with period q with respect to ϕ if $P_{\text{top}}(T, \phi) = \frac{1}{q} \sum_{h=0}^{q-1} \phi T^h(x_0)$. If x_0 is not indifferent, then we call x_0 a *generalized repelling periodic point*.

If a potential ϕ of WBV admits a generalized indifferent periodic point, then we can observe interesting statistical phenomena. More specifically, if there exists an $\exp[P_{\text{top}}(T, \phi) - \phi]$ -conformal measure ν , then the above definitions can be described in terms of the local Jacobians with respect to ν , that is,

$$\frac{d(\nu T^q)}{d\nu}|_{X_{i_1 \dots i_q}(x_0)}(x_0) = \exp[qP_{\text{top}}(T, \phi) - \sum_{h=0}^{q-1} \phi T^h(x_0)] = 1.$$

Then we have the following facts.

Proposition 3. *Let x_0 be a generalized indifferent periodic point with period q with respect to $\phi \in \mathcal{W}(T)$. Let ν be an $\exp[P_{\text{top}}(T, \phi) - \phi]$ -conformal measure. Then*

- (i) $\forall s \geq 1, P_{\text{top}}(T, s\phi) = sP_{\text{top}}(T, \phi)$ and $\forall s < 1, P_{\text{top}}(T, s\phi) \geq sP_{\text{top}}(T, \phi)$.
- (ii) $\nu(X_{i_1 \dots i_n}(x_0))$ decays subexponentially fast.

Proof. By Lemma 10, we have $P_{\text{top}}(T, s\phi) \geq s \frac{1}{q} \sum_{i=0}^{q-1} \phi T^i(x_0)$. In particular, if x_0 is a generalized indifferent periodic point for ϕ , then $P_{\text{top}}(T, s\phi) \geq sP_{\text{top}}(T, \phi)$. We recall that by Lemma 2 the conformal measure ν is a weak Gibbs measure for ϕ of WBV. Then we have for $s \geq 1$,

$$\frac{1}{n} \log Z_n(s\phi) \leq sP_{\text{top}}(T, \phi) + \frac{1}{n} \log C_n K_n^s + \frac{1}{n} \log \sum_{i_1 \dots i_n} \nu(X_{i_1 \dots i_n})^s,$$

where both C_n and K_n satisfy $\lim_{n \rightarrow \infty} \frac{1}{n} \log C_n = 0$ and $\lim_{n \rightarrow \infty} \frac{1}{n} \log K_n = 0$. Since $\lim_{n \rightarrow \infty} \frac{1}{n} \log C_n K_n^s = 0$, we have $P_{\text{top}}(T, s\phi) \leq sP_{\text{top}}(T, \phi)$ for $s \geq 1$. (ii) follows from Proposition 6.1 in [21]. \square

Let us recall that ν was obtained by constructing a jump transformation in Theorem 4. Then we can associate the generalized indifferent periodic points to the marginal sets $\bigcap_{n \geq 0} D_n$.

Proposition 4. *Let x_0 be a generalized indifferent periodic point with period q with respect to $\phi \in \mathcal{W}(T)$.*

- (i) (Failure of bounded distortion)

$$C_{nq}(x_0) := \sup_{x, y \in X_{i_1 \dots i_{nq}}(x_0)} \frac{\exp[\sum_{h=0}^{nq-1} \phi T^h(x)]}{\exp[\sum_{h=0}^{nq-1} \phi T^h(y)]} \rightarrow \infty$$

monotonically as $n \rightarrow \infty$.

- (ii) $x_0 \in \bigcap_{n \geq 0} D_n$.

Proof. Since $C_n(x_0)$ is the distortion of $\frac{d(\nu T^n)}{d\nu}$ over cylinders $X_{i_1 \dots i_n}(x_0)$, (i) follows from Lemma 6.1 in [21]. Suppose $x_0 \notin \bigcap_{n \geq 0} D_n$. Then by Sublemma A (see §9) we have $x_0 \in X^*$. Since $\sum_{n=1}^{\infty} \text{Var}_n(T^*, \phi^*) < \infty$ implies that $C_{nq}(x_0)$ cannot increase monotonically, we have a contradiction to (i). We complete the proof. \square

Remark (C). We claim that $\bigcap_{n \geq 0} D_n$ can contain repelling periodic points.

If we have a T -invariant probability measure μ equivalent to ν via Kac's formula (Lemma 16) or Schweiger's formula (3) in §5, then the invariant densities $d\mu/d\nu$ are typically unbounded at indifferent periodic points with respect to ν (Lemma 6.2 in [21]) so that we can see interesting phenomena from a statistical point of view ([19], [21]). For example, under the existence of a generalized indifferent periodic point x_0 with respect to ϕ , the rate of decay of correlation may be slower than $\nu(X_{i_1 \dots i_n}(x_0))$, which decays subexponentially fast by (ii) in Proposition 3. We referee [21] for further details. On the other hand, the Dirac measure m supported on the generalized indifferent periodic orbit with respect to ϕ satisfies $P_{\text{top}}(T, \phi) = h_m(T) + \int_X \phi dm$. Hence if we can establish a variational principle for the topological pressure and can construct a T -invariant measure μ equivalent to the weak Gibbs measure ν for ϕ with $-P_{\text{top}}(T, \phi)$, then by Lemma 3 we see immediately failure of uniqueness of equilibrium states. Furthermore, by the definition of *indifferency* we can show failure of Gibbsianness of equilibrium states for ϕ with generalized indifferent periodic points.

Theorem 5 (Characterization of non-Gibbsianness). *Suppose that a potential ϕ with $P_{\text{top}}(T, \phi) < \infty$ admits a generalized indifferent periodic point x_0 . Then there is no Borel probability measure that is Gibbs for ϕ .*

§5. EQUILIBRIUM STATES FOR POTENTIALS OF WEAK BOUNDED VARIATION

Let (T, X, Q) be a transitive FRS Markov system and let $M_T(X)$ be the set of all T -invariant probability measures on (X, \mathcal{F}) . For $m \in M_T(X)$, I_m denotes the conditional information of Q with respect to $T^{-1}\mathcal{F}$. We denote

$$M_T(X, \phi) := \{m \in M_T(X) \mid I_m + \phi \in L^1(m), \text{ either } h_m(T) < \infty \text{ or } \int_X \phi dm > -\infty \text{ is satisfied}\}.$$

Theorem 6. *Let (T, X, Q) be a transitive FRS Markov system and let $\phi \in \mathcal{W}(T)$. Suppose that there exists a union of full cylinders $B_1(\subset X)$ with respect to which (T, X, Q) satisfies local exponential instability and ϕ satisfies local bounded distortion. Let ν be the $\exp[P_{\text{top}}(T, \phi) - \phi]$ -conformal measure supported on X^* . Assume further that $\Gamma := \bigcap_{n \geq 0} D_n$ consists of periodic points. If $\int_{X^*} R d\nu < \infty$ and $H_\nu(Q^*) < \infty$, then there exists a T -invariant ergodic probability measure μ equivalent to ν that satisfies the following variational principle:*

$$P_{\text{top}}(T, \phi) = h_\mu(T) + \int_X \phi d\mu = \sup\{h_m(T) + \int_X \phi dm \mid m \in M_T(X, \phi) \text{ is ergodic}\}.$$

If $E_T(X, \phi) := \{m \in M_T(X, \phi) \mid h_m(T) + \int_X \phi dm = P_{\text{top}}(T, \phi)\}$ contains at least two elements, then it implies physically coexistence of different phases, which is so-called “*phase transition*”. Phase transition may be related to failure of the Gibbs property of equilibrium states (see Theorem 5).

Corollary 2 (Phase transition). *We assume all conditions in Theorem 6. If Γ consists of generalized indifferent periodic points with respect to ϕ , then the set of equilibrium states for ϕ is the convex hull of μ and the set of invariant Borel probability measures supported on Γ .*

In order to prove Theorem 6, we need a sequence of lemmas. Let $M_{T^*}(X^*)$ denote the set of all Borel probability measures on X^* invariant under T^* . For $\phi^* - sR$ we define

$$M_{T^*}(X^*, \phi^* - sR) := \{m^* \in M_{T^*}(X^*) \mid \text{either } h_{m^*}(T^*) < \infty \text{ or} \\ \int_{X^*} (\phi^* - sR) dm^* > -\infty \text{ is satisfied}\}.$$

Let $s_0 = P_{\text{top}}(T, \phi)$. Then P. Walter's method in [14] can apply for T^* and for $\phi^* - s_0R$ so that there exists the unique equilibrium state μ^* equivalent to ν and the following variational principle is valid:

$$0 = P_{\text{top}}(T^*, \phi^* - s_0R) = h_{\mu^*}(T^*) + \int_{X^*} (\phi^* - s_0R) d\mu^* \\ = \sup\{m^* \in M_{T^*}(X^*, \phi^* - s_0R) \mid h_{m^*}(T^*) + \int_{X^*} (\phi^* - s_0R) dm^*\}.$$

Since $\sum_{n=1}^{\infty} \text{Var}_n(T^*, \phi^* - s_0R) < \infty$ implies the bounded distortion property with respect to ν :

$$\sup_{n \geq 1} \sup_{Y \in \mathcal{V}_{j=0}^{n-1}} \sup_{T^{*-j}(Q^*)} \sup_{x, x' \in Y} \frac{\frac{d(\nu T^{*n})}{d\nu}|_Y(x)}{\frac{d(\nu T^{*n})}{d\nu}|_Y(x')} < \infty,$$

we can show ergodicity and Bowen's Gibbs property for μ^* . If $\int_{X^*} R d\mu^* < \infty$, then the next Schweiger's formula ([13]) gives a T -invariant ergodic probability measure μ equivalent to ν that satisfies $\mu(B_1) = (\int_{X^*} R d\mu^*)^{-1} > 0$:

$$(3) \quad \left(\int_{X^*} R d\mu^*\right) \mu(E) = \sum_{n=0}^{\infty} \mu^*(D_n \cap T^{-n}E) = \int_{X^*} \sum_{i=0}^{R(x)-1} 1_E T^i(x) d\mu^*(x)$$

and by Lemma 5 for $f \in L^1(\mu)$,

$$(3)^* \quad \int_X f d\mu = \frac{\int_{X^*} \sum_{i=0}^{R(x)-1} f T^i(x) d\mu^*}{\int_{X^*} R d\mu^*} = \frac{\int_{B_1} \sum_{i=0}^{R_{B_1}(x)-1} f T^i(x) d\mu_{B_1}}{\int_{B_1} R_{B_1} d\mu_{B_1}}$$

(cf. Lemma 4.2 in [18]). Since $\int_{X^*} R d\mu^* < \infty$ gives the equality

$$\int_{X^*} (\phi^* - s_0R) d\mu^* = \int_{X^*} \phi^* d\mu^* - s_0 \int_{X^*} R d\mu^*$$

and $H_{\mu^*}(Q^*) < \infty$ gives $h_{\mu}(T) < \infty$, we can establish the following characterization of the zero s_0 of (GBE).

Lemma 11. *If $\mu^* \in M_{T^*}(X^*)$ is ergodic and satisfies $h_{\mu^*}(T^*) + \int_{X^*} (\phi^* - s_0R) d\mu^* = 0$, $\int_{X^*} R d\mu^* < \infty$ and $H_{\mu^*}(Q^*) < \infty$, then*

$$s_0 = \frac{h_{\mu^*}(T^*) + \int_{X^*} \phi^* d\mu^*}{\int_{X^*} R d\mu^*} = h_{\mu}(T) + \int_X \phi d\mu,$$

where μ is obtained by formula (3).

By Lemma 11 we have a T -invariant ergodic probability measure μ equivalent to ν that satisfies $P_{\text{top}}(T, \phi) = h_\mu(T) + \int_X \phi d\mu$.

Lemma 12 (Lemma 4.4 in [18]). *If a T -invariant probability measure m satisfies $m(B_1) = 0$, then $\Gamma := \bigcap_{n \geq 0} D_n$ is a full measure set with respect to m .*

Proof of Theorem 6. By Lemma 11 for all T -invariant ergodic probability measures m on X with $m(B_1) > 0$ and $m \in M_T(X, \phi)$, we can establish

$$0 = \frac{h_\mu(T) + \int_X \phi d\mu - P_{\text{top}}(T, \phi)}{\mu(B_1)} \geq \frac{h_m(T) + \int_X \phi dm - P_{\text{top}}(T, \phi)}{m(B_1)}.$$

On the other hand, by Lemma 12, any T -invariant ergodic probability measure m on X with $m(B_1) = 0$ satisfies $m(\Gamma) = 1$. In particular, if $\Gamma := \bigcap_{n \geq 0} D_n$ consists of periodic points, then $h_m(T) + \int_X \phi dm = \int_\Gamma \phi dm \leq P_{\text{top}}(T, \phi)$, which completes the proof of Theorem 6. \square

§6. THE CONSTRUCTION OF σ -FINITE CONFORMAL MEASURES VIA INDUCED MAPS

Let (T, X, Q) be a transitive FRS Markov system and $\phi \in \mathcal{W}(T)$. Suppose that (T, X, Q) satisfies local exponential instability and ϕ satisfies local bounded distortion with respect to $B_1 = \bigcup_{j \in J} X_j (J \subset I)$. Let $A_j = \text{cl}(\text{int} X_j)$ for $j \in J$ and put $A := \bigcup_{j \in J} A_j$. We define the first return function $R_A : A \rightarrow \mathbb{R} \cup \{\infty\}$ and the induced map T_A over $\{x \in A : R_A(x) < \infty\}$. By the Markov property, there exists a partition of the set $B_k^{(A)} = \{x \in A : R_A(x) = k\}$ for each $k \geq 1$ so that T^k restricted to the interior of each element of the partition is a homeomorphism onto its image. I_A denotes the set of all indices corresponding to such elements of the partition of $\bigcup_{k \geq 1} B_k^{(A)}$. Then $\{v_{\underline{i}} : \underline{i} \in I_A\}$ is a family of extensions of local inverses of T_A . For $s \in \mathbb{R}$ and $x \in \bigcup_{k=1}^\infty B_k^{(A)}$ we define $\phi_A^{(s)}(x) = \sum_{h=0}^{R_A(x)-1} \phi T^h(x) - s R_A(x)$. By Lemma 4 we can easily see the next fact.

Lemma 13. *If each $A_i \subset A$ satisfies $T A_i = X$, then*

$$P_{\text{top}}(T^*, \phi^* - sR) = P_{\text{top}}(T_A, \phi_A^{(s)}).$$

We recall the following result in [6].

Lemma 14 ([6]). *If $\|\mathcal{L}_{\phi_A^{(0)}} 1\| < \infty$, then $s \rightarrow P_{\text{top}}(T_A, \phi_A^{(s)})$ is continuous on $\text{int}\{s \in \mathbb{R} : P_{\text{top}}(T_A, \phi_A^{(s)}) \in \mathbb{R}\}$.*

We suppose that B_1 consists of a single full cylinder X_j and the following conditions are satisfied for $A = \text{cl}(\text{int} X_j)$.

(05)* $\exists 0 < \gamma < 1, 0 < \overline{\gamma} < \infty$ such that

$$\sigma_{A, T_A}(n) = \sup\{\text{diam } Y \mid Y \in \bigvee_{j=0}^{n-1} T_A^{-j}(Q_A)\} \leq \overline{\gamma} \gamma^n$$

and there exists $\theta > 0$ such that

(06)* $\forall \underline{i} = (i_1 \dots i_{|\underline{i}|}) \in I_A$ and all $0 \leq j < |\underline{i}|$, $\exists 0 < L_\phi(i_{j+1} \dots i_{|\underline{i}|}) < \infty$ satisfying

$$|\phi(v_{i_{j+1} \dots i_{|\underline{i}|}}(x)) - \phi(v_{i_{j+1} \dots i_{|\underline{i}|}}(y))| \leq L_\phi(i_{j+1} \dots i_{|\underline{i}|}) d(x, y)^\theta \quad (\forall x, y \in A)$$

and

$$\sup_{\underline{i} \in I_A} \sum_{j=0}^{|\underline{i}|-1} L_\phi(i_{j+1} \dots i_{|\underline{i}|}) < \infty.$$

Since the conditions (05-06)* allow us to establish the WBV property of $\phi_A^{(s)}$, by Theorem 1, $\exists \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n(\phi_A^{(s)}) := P_{\text{top}}(T_A, \phi_A^{(s)})$, where

$$Z_n(\phi_A^{(s)}) = \sum_{(\underline{i}_1 \dots \underline{i}_n) \in I_A^n} \sum_{v_{\underline{i}_1} \dots v_{\underline{i}_n} x = x} \exp\left[\sum_{m=0}^{n-1} \phi_A^{(s)} T_A^m(x)\right].$$

Furthermore, (05-06)* guarantee equi-Hölder continuity of $\{\phi_A^{(s)} v_{\underline{i}} : \underline{i} \in I_A\}$ and $\sum_{n=1}^{\infty} \text{Var}_n(T_A, \phi_A^{(s)}) < \infty$. Hence if $\|\mathcal{L}_{\phi_A^{(s)}} 1\| < \infty$, then $\mathcal{L}_{\phi_A^{(s)}} : C(A) \rightarrow C(A)$.

Theorem 7 (A construction of σ -finite conformal measures via induced maps). *Let (T, X, Q) be a transitive FRS Markov system and let $\phi \in \mathcal{W}(T)$. Suppose that there exists a full cylinder $X_j \in Q$ satisfying (05)* and (06)* for $A := cl(int X_j)$. If*

$$\|\mathcal{L}_{\phi_A^{(0)}} - R_A \min\{0, P_{\text{top}}(T_A, \phi_A^{(0)})\} 1\| < \infty,$$

then

- (i) $\exists s_0 \in \mathbb{R}$ with $P_{\text{top}}(T_A, \phi_A^{(s_0)}) = 0$ (a generalized Bowen's equation);
- (ii) there exists a Borel probability measure ν_A on $(A, \mathcal{F} \cap A)$ with $\nu_A(\{x \in A \mid R_A(x) < \infty\}) = 1$ satisfying $\mathcal{L}_{\phi_A^{(s_0)}}^* \nu_A = \nu_A$;
- (iii) there exists a σ -finite measure ν on X satisfying $\mathcal{L}_\phi^* \nu = [\exp s_0] \nu$ and $\nu(\bigcup_{i \in I} \partial X_i) = 0$;
- (iv) in particular, if ν is finite, then $P_{\text{top}}(T, \phi) = s_0$.

Proof of Theorem 7. By Lemmas 13-14 and Proposition 1, we have the existence of $s \in \mathbb{R}$ for which $P_{\text{top}}(T_A, \phi_A^{(s)}) = 0$ and $\mathcal{L}_{\phi_A^{(s)}}^*$ admits an eigenvalue 1. Then we can apply the main theorem in [5] so that (i)-(iv) are obtained. \square

The next result gives a criterion of finiteness of ν .

Proposition 5 (A criterion of finiteness of ν). *Suppose that all assumptions in Theorem 7 are satisfied. Then $\nu(X) = \int_A \exp[s_0 - \phi] d\nu_A + 1$. In particular, if $\inf_{x \in A} \phi(x) > -\infty$, then ν is finite.*

Proof of Proposition 5. Let $I' = \{\underline{i} \mid X_{\underline{i}} \subset D_{|\underline{i}|}\}$: First we note the following formula of ν , which was obtained in [5]:

$$\nu(E) = \sum_{\underline{i} \in I'} \int_A 1_E(v_{\underline{i}} x) \exp\left[\sum_{l=0}^{|\underline{i}|-1} \phi T^l(v_{\underline{i}} x) - s_0 |\underline{i}|\right] d\nu_A(x) + \int_A 1_E(x) d\nu_A(x).$$

Then we see that $\nu(X)$ is equal to

$$\sum_{k=1}^{\infty} \sum_{\underline{j} \in I_A, |\underline{j}|=k+1} \int_{T_A v_{\underline{j}}(A)} \exp\left[\sum_{l=1}^k (\phi T^l - s_0)(v_{\underline{j}} x)\right] d\nu_A(x) + 1$$

because of the fact that $X_{\underline{i}} \subset TA(\forall \underline{i} \in I')$. By conformality of ν_A this coincides with

$$\sum_{k=1}^{\infty} \sum_{\underline{j} \in I_A, |\underline{j}|=k+1} \int_{v_{\underline{i}}(A)} \exp\left[\left(\sum_{l=1}^k \phi T^l - s_0\right)(x)\right] \exp[-\phi_A^{(s_0)}(x)] d\nu_A(x) + 1.$$

□

§7. THE CONSTRUCTION OF MUTUALLY SINGULAR EQUILIBRIUM STATES

In this section, we show the existence of mutually singular non-atomic equilibrium states by using induced systems.

Lemma 15. *Let (T, X, Q) be a transitive FRS Markov system and let $\phi \in \mathcal{W}(T)$. Suppose that there exists a sequence of full cylinders $\{X_i\}_{i=1}^M$ ($M \leq \infty$) that satisfies $(05)^*$ and $(06)^*$, $\inf_{x \in A_i} \phi(x) > -\infty$, and*

$$\|\mathcal{L}_{\phi_{A_i}^{\min\{0, P_{\text{top}}(T_{A_i}, \phi_{A_i}^{(0)})\}}} 1\| < \infty$$

for each $A_i = cl(int X_i)$. Let $\Gamma_0 := X$ and for each $i \geq 0$ define inductively $\Gamma_{i+1} = \bigcap_{n=0}^{\infty} T^{-n}(\Gamma_i \cap A_{i+1}^c) (\subset \Gamma_i)$. We assume that for each $i \geq 0$,

$$H_{\nu_{\Gamma_i \cap A_{i+1}}}(Q_{\Gamma_i \cap A_{i+1}}) < \infty, \quad \int_{\Gamma_i \cap A_{i+1}} R_{\Gamma_i \cap A_{i+1}} d\nu_{\Gamma_i \cap A_{i+1}} < \infty$$

for the Borel probability measure $\nu_{\Gamma_i \cap A_{i+1}}$ on $\Gamma_i \cap A_{i+1}$ obtained in Theorem 7. If $\bigcap_{i=1}^M \Gamma_i := \Gamma$ consists of periodic points, then there exists a T -invariant ergodic probability measure μ equivalent to an $\exp[P_{\text{top}}(T, \phi) - \phi]$ -conformal measure ν that satisfies the following variational principle:

$$P_{\text{top}}(T, \phi) = h_{\mu}(T) + \int_X \phi d\mu = \sup\{h_m(T) + \int_X \phi dm \mid m \in M_T(X, \phi) \text{ is ergodic}\}.$$

The equilibrium state μ for ϕ is not necessarily unique.

Theorem 8 (Phase transition and singular equilibrium states). *We assume all conditions in Lemma 15. If $\Gamma := \bigcap_{i=1}^M \Gamma_i$ consists of generalized indifferent periodic points with respect to ϕ , then there exists a sequence of ergodic equilibrium states $\{\mu_i\}_{i=1}^M$ that are mutually singular and the set of equilibrium states for ϕ is the convex hull of $\{\mu_i\}_{i=1}^M$ and the set of invariant Borel probability measures supported on Γ .*

Lemma 16 (Kac's formula). *If $\int_A R_A d(\nu|_A) < \infty$ and μ_A is a T_A -invariant ergodic probability measure equivalent to $\nu|_A$, then the next formula gives a T -invariant ergodic probability measure μ equivalent to ν :*

$$\mu(E)/\mu(A) = \int_A \sum_{i=0}^{R_A(x)-1} 1_E \circ T^i(x) d\mu_A(x) (\forall E \in \mathcal{F}).$$

Lemma 17 (Finite entropy condition). *Suppose that for $s_i := P_{\text{top}}(T|_{\Gamma_{i-1}}, \phi)$, $\int_{\Gamma_{i-1} \cap A_i} \phi_{\Gamma_{i-1} \cap A_i}^{(s_i)} d\nu_{\Gamma_{i-1} \cap A_i} > -\infty$. Then $H_{\nu_{\Gamma_i \cap A_{i+1}}}(Q_{\Gamma_i \cap A_{i+1}}) < \infty$.*

Proof of Lemma 17. Let $A = \Gamma_i \cap A_{i+1}$ and $s = s_i$. Since we have $\forall \underline{j} \in I_A$,

$$\nu_A(X_{\underline{j}}) = \int_A \exp[\phi_A^{(s)}(v_{\underline{j}}(x))] d\nu_A(x) \geq \exp[-s|\underline{j}|] \inf_{x \in v_{\underline{j}}(A)} \exp[\phi_A^{(0)}(x)],$$

the bounded distortion for $\phi_A^{(0)}$ allows us to see that

$$\begin{aligned} \int_A (-\phi_A^{(0)}) d\nu_A &\geq -\log C - \sum_{\underline{j} \in I_A} \int_{X_{\underline{j}}} \log(\inf_{x \in v_{\underline{j}}(A)} \exp[\phi_A^{(0)}(x)]) d\nu_A(x) \\ &\geq -\log C + \sum_{\underline{j} \in I_A} \int_{X_{\underline{j}}} (-s|\underline{j}|) d\nu_A - \sum_{\underline{j} \in I_A} \nu_A(X_{\underline{j}}) \log(\nu_A(X_{\underline{j}})), \end{aligned}$$

where C is the bounded distortion constant $\exp[D\bar{\gamma}(1-\gamma)^{-1}(\text{diam } X)^\theta]$. These inequalities allow one to establish

$$H_{\nu_A}(Q_A) \leq \int_A (-\phi_A^{(0)}) d\nu_A + \log C + s \sum_{\underline{j} \in I_A} \int_{X_{\underline{j}}} R_A d\nu_A < \infty. \quad \square$$

Proof of Lemma 15. Since $\sum_{n=1}^{\infty} \text{Var}_n(T_{A_1}, \phi_{A_1}^{(s)}) < \infty$ is satisfied, it follows from Theorem 7 that $\exists \nu_{A_1}$ on A_1 satisfying $\mathcal{L}_{\phi_{A_1}^{(s)}}^* \nu_{A_1} = \nu_{A_1}$ for $s = P_{\text{top}}(T, \phi)$ and $\exists \nu$ an $\exp[P_{\text{top}}(T, \phi) - \phi]$ -conformal measure on X . Furthermore, by Proposition 5 we see that $\frac{\nu|_{A_1}}{\nu(A_1)} = \nu_{A_1}$. The bounded distortion allows one to obtain an ergodic T_{A_1} -invariant probability measure $\mu_{A_1} \sim \nu_{A_1}$ with a density $d\mu_{A_1}/d\nu_{A_1}$ away from zero and infinity. Furthermore, by [14] there exists an equilibrium state μ_{A_1} for $\phi_{A_1}^{(s)}$ with respect to T_{A_1} that is ergodic. In particular, since $H_{\nu_{A_1}}(Q_{A_1}) < \infty$ we have for $s_1 = P_{\text{top}}(T, \phi)$,

$$P_{\text{top}}(T_{A_1}, \phi_{A_1}^{(s_1)}) = 0 = h_{\mu_{A_1}}(T_{A_1}) + \int_{A_1} \phi_{A_1}^{(s_1)} d\mu_{A_1} \geq h_{m_{A_1}}(T_{A_1}) + \int_{A_1} \phi_{A_1}^{(s_1)} dm_{A_1}$$

for all T_{A_1} -invariant probability measures $m_{A_1} \in M_{T_{A_1}}(A_1, \phi_{A_1}^{(s_1)})$ (cf. [14]). These inequalities and Lemma 16 allow us to have a T -invariant ergodic probability measure $\mu_1 \sim \nu$ that satisfies $\mu_1(A_1) > 0$, $\mu_1(\Gamma_1) = 0$ and

$$\begin{aligned} (**) \quad 0 &= \mu_1(A_1)^{-1} (h_{\mu_1}(T) + \int_X (\phi - s_1) d\mu_1) \\ &\geq m(A_1)^{-1} (h_m(T) + \int_X (\phi - s_1) dm) \end{aligned}$$

for all T -invariant ergodic probability measures $m \in M_T(X, \phi)$ satisfying $m(A_1) > 0$. (**) is equivalent to the inequalities : $s_1 = h_{\mu_1}(T) + \int_X \phi d\mu_1 \geq h_m(T) + \int_X \phi dm$. On the other hand, any $m \in M_T(X, \phi)$ satisfying $m(A_1) = 0$ is supported on Γ_1 . In fact, since $X = (\bigcup_{i=0}^{\infty} T^{-i} A_1) \cup (\bigcap_{i=0}^{\infty} (T^{-i} A_1)^c)$, $m(A_1) = 0$ and T -invariance of m give $m(\Gamma_1) = 1$. Thus the set of all T -invariant probability measures m supported on Γ_1 coincides with the set of all T -invariant probability measures m with $m(A_1) = 0$. Since $(T|_{\Gamma_1}, \Gamma_1, Q_{\Gamma_1} := Q \cap \Gamma_1)$ is a subsystem of (T, X, Q) , we can apply the above arguments for the induced system $(T_{A_2 \cap \Gamma_1}, A_2 \cap \Gamma_1, Q_{A_2 \cap \Gamma_1})$. That is, for $s_2 = P_{\text{top}}(T|_{\Gamma_1}, \phi) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\underline{i}: |\underline{i}|=n} \sum_{v_{\underline{i}} x = x, x \in \Gamma_1} \exp[\sum_{h=0}^{n-1} \phi T^h(x)]$ and for the associated potential $\phi_{A_2 \cap \Gamma_1}^{(s_2)}(x) = \sum_{h=0}^{R_{A_2 \cap \Gamma_1}(x)-1} \phi T^h(x) - s_2 R_{A_2 \cap \Gamma_1}(x)$

our assumptions allow us to establish the bounded distortion for $\phi_{A_2 \cap \Gamma_1}^{(s_2)}$ so that there exists a $T_{A_2 \cap \Gamma_1}$ -invariant ergodic probability measure $\mu_{A_2 \cap \Gamma_1}$ that satisfies

$$\begin{aligned} 0 &= h_{\mu_{A_2 \cap \Gamma_1}}(T_{A_2 \cap \Gamma_1}) + \int_{A_2 \cap \Gamma_1} \phi_{A_2 \cap \Gamma_1}^{(s_2)} d\mu_{A_2 \cap \Gamma_1} \\ &\geq h_{m_{A_2 \cap \Gamma_1}}(T_{A_2 \cap \Gamma_1}) + \int_{A_2 \cap \Gamma_1} \phi_{A_2 \cap \Gamma_1}^{(s_2)} dm_{A_2 \cap \Gamma_1} \end{aligned}$$

for all $T_{A_2 \cap \Gamma_1}$ -invariant ergodic probability measures $m_{A_2 \cap \Gamma_1} \in M_{T_{A_2 \cap \Gamma_1}}(A_2 \cap \Gamma_1, \phi_{A_2 \cap \Gamma_1}^{(s_2)})$. Let μ_2 be the T -invariant ergodic probability measure supported on Γ_1 arising from $\mu_{A_2 \cap \Gamma_1}$ via Kac's formula. Then $s_2 = h_{\mu_2}(T) + \int_X \phi d\mu_2 \geq h_m(T) + \int_X \phi dm$ for all T -invariant ergodic probability measures $m \in M_T(X, \phi)$ supported on Γ_1 that satisfy $m(\Gamma_1 \cap A_2) > 0$. Inductively we have a decreasing sequence $\{s_i\}_{i \in \mathbb{N}}$, where $s_i = P_{\text{top}}(T|_{\Gamma_{i-1}}, \phi)$ and a sequence of T -invariant ergodic probability measures $\{\mu_i\}_{i=1}^M$ such that μ_i is supported on Γ_{i-1} , $\mu_i(\Gamma_{i-1} \cap A_i) > 0$ and s_i satisfies $s_i = h_{\mu_i}(T) + \int_X \phi d\mu_i \geq h_m(T) + \int_X \phi dm$ for all T -invariant ergodic probability measures $m \in M_T(X, \phi)$ supported on Γ_{i-1} with $m(\Gamma_{i-1} \cap A_i) > 0$. Since $\mu_i(\Gamma_{i-1}) = 1$ and $\mu_i(\Gamma_i) = 0$, $\{\mu_i\}_{i=1}^M$ are mutually singular. Finally, for every T -invariant measure supported on Γ that consists of periodic points we have $s_1 \geq h_m(T) + \int_X \phi dm$. Since $\{s_i\}_{i \in \mathbb{N}}$ is decreasing, we complete the proof. \square

§8. APPLICATIONS

In this section, we show some examples of transitive FRS Markov systems to which our theorems 1-8 can apply.

Example 1 (Brun's map [13], [18], [20]). Let $X = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_2 \leq x_1 \leq 1\}$, and let

$$X_i = \{(x_1, x_2) \in X : x_i + x_1 \geq 1 \geq x_{i+1} + x_1\}$$

for $i = 0, 1, 2$ where we put $x_0 = 1$ and $x_3 = 0$. T is defined by

$$\begin{aligned} T(x_1, x_2) &= \left(\frac{x_1}{1-x_1}, \frac{x_2}{1-x_1}\right) \text{ on } X_0, \\ T(x_1, x_2) &= \left(\frac{1}{x_1} - 1, \frac{x_2}{x_1}\right) \text{ on } X_1, \\ T(x_1, x_2) &= \left(\frac{x_2}{x_1}, \frac{1}{x_1} - 1\right) \text{ on } X_2. \end{aligned}$$

Then $Q = \{X_i\}_{i=0}^2$ is a Bernoulli partition and $(0, 0)$ is an indifferent fixed point (i.e., $|\det DT(0, 0)| = 1$). Since T is a continuous piecewise C^2 map and $\sigma(n) = n^{-1}$, all conditions (01)-(04) are satisfied and dynamical instability is polynomial. We see that $\phi = -\log |\det DT|$ is piecewise Lipschitz continuous so that ϕ is a potential of WBV. Furthermore, since each periodic point is contained in a single cylinder the property (1) is satisfied. Define $B_1 = X_1 \cup X_2$. Then T^* satisfies the uniformly expanding property and a direct calculation allows us to establish (06) for $\phi = -\log |\det DT|$ (see [18] for more details). Hence we can apply Theorems 1-8. In particular, we can see that $P_{\text{top}}(T, \phi) = P_{\text{top}}(T^*, \phi^*) = 0$.

Example 2 (Inhomogeneous Diophantine approximations [13], [15], [16], [17], [19], [20], [21]). For the transformation defined in the introduction (Example A), we can directly verify all conditions (01)-(04). In fact, we can introduce an index set

$$I = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} : a, b \in \mathbb{Z}, a > b > 0, \text{ or } a < b < 0 \right\}$$

and a partition $\left\{ X_{\begin{pmatrix} a \\ b \end{pmatrix}} : \begin{pmatrix} a \\ b \end{pmatrix} \in I \right\}$, where $X_{\begin{pmatrix} a \\ b \end{pmatrix}} = \{(x, y) \in X : a = [\frac{1-y}{x}] - [-\frac{y}{x}], b = -[-\frac{y}{x}]\}$. Although $\phi = -\log |\det DT|$ fails (piecewise) Hölder continuity,

we can verify $V_n(\phi) \leq \log(1 + \sigma(n-2))$ and $\sigma(n) = O(n^{-1})$ (see [15], [20]) (cf. [16], [19], [21]). Hence $\phi = -\log |\det DT|$ is a potential of WBV. Since each periodic point is contained in a single cylinder, the property (1) in Lemma 6 is satisfied. Let D_n be the union of cylinders of rank n containing indifferent periodic points and let $B_n = D_{n-1} \setminus D_n$. Then the jump transformation $T^* : \bigcup_{i=1}^{\infty} B_i \rightarrow X$ defined by $T^*(x, y) = T^i(x, y)$ for $(x, y) \in B_i$ satisfies exponential decay of diameter of cylinders (see [19]), and we can verify the validity of (06) for $\phi = -\log |\det DT|$. Indeed, for $\underline{i} \in I^*$ with $|\underline{i}| = n$, $L_\phi(\underline{i}) \leq 3/n^2$ and so

$$\sup_{\underline{i} \in I^*} \sum_{j=0}^{|\underline{i}|-1} L_\phi(i_{j+1} \dots i_{|\underline{i}|}) \leq \sum_{n=1}^{\infty} \frac{3}{n^2} < \infty.$$

Hence we have summability of $Var_n(T^*, \phi^*)$, which allows us to apply Theorems 1 - 8.

Example 3 (A complex continued fraction [5], [12], [15], [22]). For the transformation T , defined in the introduction (Example B), we define $X_{n\alpha+m\bar{\alpha}} = \{z \in X : [1/z]_1 = n\alpha + m\bar{\alpha}\}$ for each $n\alpha + m\bar{\alpha} \in I := \{m\alpha + n\bar{\alpha} : (m, n) \in \mathbb{Z}^2 - (0, 0)\}$. Then we have a countable partition $Q = \{X_a\}_{a \in I}$ of X that is a topologically mixing Markov partition and satisfies (01)-(03). The inverse branches to T take the form $v_j(z) = 1/(j+z)$, where $j \in I$ and the v_j satisfy (04). Therefore the inverse branches of the n th iterate of the transformation T^n take the form

$$v_{j_1, \dots, j_n}(z) = \frac{p_n + zp_{n-1}}{q_n + zq_{n-1}} \text{ and } |v'_{j_1, \dots, j_n}(z)| = \frac{1}{|q_n + zq_{n-1}|^2}$$

where $p_n = j_n p_{n-1} + p_{n-2}$ and $q_n = j_n q_{n-1} + q_{n-2}$, $n \geq 1$, and $p_{-1} = \alpha$, $p_0 = 0 = q_{-1}$ and $q_0 = \alpha$. If the string j_1, \dots, j_{n-1} corresponds to a cylinder that contains one of the indifferent points, but the longer string j_1, \dots, j_n corresponds to a sub-cylinder disjoint from the indifferent periodic points, then v_{j_1, \dots, j_n} is an inverse branch of the jump transformation T^* which is uniformly expanding. For $\phi(z) = -\log |T'(z)|$, WBV and (06) are satisfied. Further details can be found in [22] in which multifractal formalism was established by applying our Theorems 1-8.

§9. APPENDIX - PROOFS OF RESULTS IN §§2 AND 3

For the proof of Theorem 1, we first verify the following facts.

Lemma 18. (18-1) $\forall U_k \in \mathcal{U}, Z_n(U_k, \phi) > 0$ for all $n > S$.

(18-2) $\forall U_k \in \mathcal{U}, \forall n, m > S, \underline{Z}_{n+m}(U_k, \phi) \geq \underline{Z}_n(U_k, \phi) \underline{Z}_m(U_k, \phi)$. (Subadditivity)

(18-3) $\forall U_k, U_l \in \mathcal{U}$,

$$\begin{aligned} \underline{Z}_{n+s_l+s_k}(U_l, \phi) &\geq \underline{Z}_n(U_k, \phi) (C_{s_l} C_{s_k})^{-1} \\ &\times \sup_{x \in X^{(l,k)}(s_k)} \exp\left[\sum_{h=0}^{s_k-1} \phi T^h(x)\right] \sup_{x \in X^{(k,l)}(s_l)} \exp\left[\sum_{h=0}^{s_l-1} \phi T^h(x)\right]. \end{aligned}$$

Proof of Lemma 18. (18-1) follows from Lemma 1 and Remark (A). (18-2) follows from the definition of $\underline{Z}_n(U, \phi)$ directly. Since $\underline{Z}_{n+s_l+s_k}(U_l, \phi)$ is bounded from below by

$$\sum_{\underline{i}: |\underline{i}|=s_k, \text{int}(TX_{i_{s_k}})=U_k, \text{int}X_{i_1} \subset U_l} \inf_{x \in X_{\underline{i}}} \exp\left[\sum_{h=0}^{s_k-1} \phi T^h(x)\right]$$

$$\begin{aligned} & \times \sum_{\underline{j}: |\underline{j}|=n, \text{int}(TX_{j_n})=U_k \supset \text{int} X_{j_1}} \inf_{x \in X_{\underline{j}}} \exp \left[\sum_{h=0}^{n-1} \phi T^h(x) \right] \\ & \times \sum_{\underline{t}: |\underline{t}|=s_l, \text{int}(TX_{t_{s_l}})=U_l, \text{int} X_{t_1} \subset U_k} \inf_{x \in X_{\underline{t}}} \exp \left[\sum_{h=0}^{s_l-1} \phi T^h(x) \right] \end{aligned}$$

and the transitivity condition allows one to establish

$$\sum_{\underline{i}: |\underline{i}|=s_k, \text{int}(TX_{i_{s_k}})=U_k, \text{int} X_{i_1} \subset U_l} \inf_{x \in X_{\underline{i}}} \exp \left[\sum_{h=0}^{s_k-1} \phi T^h(x) \right] \geq \inf_{x \in X^{(l,k)}(s_k)} \exp \left[\sum_{h=0}^{s_k-1} \phi T^h(x) \right]$$

and

$$\sum_{\underline{t}: |\underline{t}|=s_l, \text{int}(TX_{t_{s_l}})=U_l, \text{int} X_{t_1} \subset U_k} \inf_{x \in X_{\underline{t}}} \exp \left[\sum_{h=0}^{s_l-1} \phi T^h(x) \right] \geq \inf_{x \in X^{(k,l)}(s_l)} \exp \left[\sum_{h=0}^{s_l-1} \phi T^h(x) \right],$$

(18-3) follows from the WBV property of ϕ . \square

Proof of Theorem 1. By (18-2) in Lemma 18 and the WBV property of ϕ , both $\lim_{n \rightarrow \infty} \frac{1}{n} \log \overline{Z}_n(U, \phi)$ and $\lim_{n \rightarrow \infty} \frac{1}{n} \log \underline{Z}_n(U, \phi)$ exist for each $U \in \mathcal{U}$. Since $\underline{Z}_n(U, \phi) \leq Z_n(U, \phi) \leq \overline{Z}_n(U, \phi)$, by the WBV property of ϕ , $\lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n(U, \phi)$ also exists and all the limits coincide. By (18-3) in Lemma 18, it is obvious that the limit does not depend on U . Noting $\min_{1 \leq j \leq N} Z_n(\phi, U_j) \leq Z_n(\phi) \leq \sum_{j=1}^N Z_n(\phi, U_j)$ allows one to complete the proof. \square

In order to prove Theorem 3 we first show the next result.

Lemma 19. (19-1) $\forall V \in \mathcal{V}$ and $\forall x \in V$, $\mathcal{L}_\phi^n 1_V(x) \leq \sum_{U_l \supseteq V} \overline{Z}_n(U_l, \phi)$.

(19-2) $\forall x \in U_k \in \mathcal{U}$, $\mathcal{L}_\phi^n 1_{U_k}(x) \geq \underline{Z}_n(U_k, \phi)$.

(19-3) $Z_n(\phi) \geq C_1^{-1} C_{n-1}^{-1} \{ \min_{1 \leq k \leq N} \sup_{y \in X^{(k)}(1)} \exp \phi(y) \} \| \mathcal{L}_\phi^{n-2} 1 \|$, where $X^{(k)}(1) \in \mathcal{Q}$ satisfies $X^{(k)}(1) \subset U_k$ and $T(\text{int } X^{(k)}(1)) = \text{int } X$.

Proof of Lemma 19. We first note that $V = \bigcup_{X_j \subset V} X_j$ because of the Markov property of Q . Then for $x \in V$ the following inequalities allow us to have the assertion in (19-1):

$$\begin{aligned} \mathcal{L}_\phi^n 1_V(x) &= \sum_{j \in I: X_j \subset V} \mathcal{L}_\phi^n 1_{X_j}(x) = \sum_{j \in I: X_j \subset V} \sum_{(j i_2 \dots i_n): x \in TX_{i_n}} \exp \left[\sum_{h=0}^{n-1} \phi T^h(v_{j i_2 \dots i_n} x) \right] \\ &= \sum_{j \in I: X_j \subset V} \sum_{U_l \in \mathcal{U}: x \in U_l} \sum_{(j i_2 \dots i_n): TX_{i_n} = U_l} \exp \left[\sum_{h=0}^{n-1} \phi T^h(v_{j i_2 \dots i_n} x) \right] \\ &= \sum_{U_l \in \mathcal{U}: U_l \supset V} \sum_{(j i_2 \dots i_n): TX_{i_n} = U_l \supset V \supset X_j} \exp \left[\sum_{h=0}^{n-1} \phi T^h(v_{j i_2 \dots i_n} x) \right] \\ &\leq \sum_{U_l \in \mathcal{U}: U_l \supset V} \sum_{(j i_2 \dots i_n): TX_{i_n} = U_l \supset X_j} \exp \left[\sum_{h=0}^{n-1} \phi T^h(v_{j i_2 \dots i_n} x) \right] \\ &\leq \sum_{U_l \in \mathcal{U}: U_l \supset V} \overline{Z}_n(U_l, \phi). \end{aligned}$$

For $x \in U_k$, we have the following inequalities, which give (19-2):

$$\begin{aligned}\mathcal{L}_\phi^n 1_{U_k}(x) &= \sum_{j \in I: X_j \subset U_k} \sum_{(ji_2 \dots i_n): x \in TX_{i_n}} \exp\left[\sum_{h=0}^{n-1} \phi T^h(v_{ji_2 \dots i_n} x)\right] \\ &= \sum_{j \in I: X_j \subset U_k} \sum_{U_l \in \mathcal{U}: x \in U_l} \sum_{(ji_2 \dots i_n): TX_{i_n} = U_l} \exp\left[\sum_{h=0}^{n-1} \phi T^h(v_{ji_2 \dots i_n} x)\right] \\ &\geq \sum_{(ji_2 \dots i_n): TX_{i_n} = U_k \supset X_j} \exp\left[\sum_{h=0}^{n-1} \phi T^h(v_{ji_2 \dots i_n} x)\right] \geq Z_n(U_k, \phi).\end{aligned}$$

By the WBV property of ϕ , $Z_n(\phi)$ is bounded from below by

$$C_1^{-1} C_{n-1}^{-1} \sum_{l=1}^N \sum_{\substack{\underline{i}: |\underline{i}|=n, \\ \text{int}(TX_{i_n})=U_l \supset \text{int} X_{i_1}}} \sup_{x \in X_{i_1}} \exp[\phi(x)] \sup_{x \in TX_{i_n}} \exp\left[\sum_{h=0}^{n-2} \phi T^h v_{i_2 \dots i_n}(x)\right].$$

Then (19-3) follows from the Markov property and the strong transitivity. \square

Proof of Theorem 3. By (19-1,2) in Lemma 19, we have for $x \in V \subset U_k$,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{L}_\phi^n 1_V(x) \leq \lim_{n \rightarrow \infty} \frac{1}{n} \log N\left(\max_{1 \leq l \leq N} \overline{Z}_n(U_l, \phi)\right) = P_{\text{top}}(T, \phi)$$

and $\liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{L}_\phi^n 1_{U_k}(x) \geq P_{\text{top}}(T, \phi)$. Our assumption $U_k \in \mathcal{V}$ implies $U_k = V$. Hence we have the first assertion. The rest of the assertions follow from (19-3) in Lemma 19 immediately. \square

In order to prove Proposition 1, we need two sublemmas.

Sublemma A (Lemma 3.1 in [17]). $\bigcap_{n>0} D_n$ and X^* are positively T -invariant. Furthermore, $\bigcup_{m=1}^\infty T^{*-m}(\bigcap_{n \geq 0} D_n)$ contains no periodic points.

Proof. The result follows from the equality (4) : $R(Tx) = R(x) - 1$ ($R(x) \geq 2$). \square

Sublemma B. Define

$$\begin{aligned}\mathcal{P}_n(X, T) &:= \{x \in X \mid \exists (i_1 \dots i_n) \in I^n \text{ such that } v_{i_1 \dots i_n} x = x\}; \\ \mathcal{P}_n(X^*, T^*) &:= \{x \in X \mid \exists (\underline{i}_1 \dots \underline{i}_n) \in I^{*n} \text{ such that } v_{\underline{i}_1 \dots \underline{i}_n} x = x\}.\end{aligned}$$

Then $\forall x \in \mathcal{P}_n(X, T) \cap X^*$, $\exists 0 < l \leq n$, $\exists y \in \mathcal{P}_l(X^*, T^*)$ such that $T^j y = x$ for some $j < R(y)$ and $\sum_{m=0}^{l-1} R(T^{*m}(y)) = n$. Furthermore, $\sum_{m=0}^{l-1} (\phi^* - sR)T^{*m}(y) = \sum_{h=0}^{n-1} (\phi - s)T^h(x)$. Conversely, $\forall y \in \mathcal{P}_l(X^*, T^*)$ and $\forall 0 \leq j < R(y)$, $T^j y \in \mathcal{P}_n(X, T) \cap X^*$ and $\sum_{m=0}^{l-1} R(T^{*m}(y)) = n$.

Proof. Since $x \in \mathcal{P}_n(X, T) \cap X^*$ visits B_1 infinitely often, we can find a point $y \in \mathcal{P}_l(X^*, T^*)$ for some $l \leq n$ such that $T^j y = x$ for some $j < R(y)$ and

$\sum_{m=0}^{l-1} R(T^{*m}(y)) = n$. By the property (4) the converse is also true. Since

$$\begin{aligned} \sum_{m=0}^{l-1} (\phi^* - sR)T^{*m}(y) &= \sum_{m=0}^{l-1} \sum_{h=0}^{R(T^{*m}y)-1} \phi T^h(T^{*m}y) - sn \\ &= \sum_{h=0}^{\sum_{m=0}^{l-1} R(T^{*m}y)-1} (\phi - s)T^h(y) \end{aligned}$$

we have the rest of the assertion. \square

Proof of Proposition 1. By Sublemma A we first note that $\mathcal{P}_n(X, T) = \{\mathcal{P}_n(X, T) \cap \bigcap_{n \geq 0} D_n\} \cup \{\mathcal{P}_n(X, T) \cap X^*\}$. Then we see that $\zeta_{T, \phi}(\exp(-s))$ is equal to

$$\zeta_{T|_{\bigcap_{n \geq 0} D_n}, \phi}(\exp(-s)) \exp\left[\sum_{n=1}^{\infty} \frac{\exp[-ns]}{n} \times \sum_{x \in \mathcal{P}_n(X, T) \cap X^*} \exp\left[\sum_{h=0}^{n-1} \phi T^h(x)\right]\right].$$

Define for $n \geq l > 0$ $E_{n,l}^* := \{y \in \mathcal{P}_l(X^*, T^*) : \sum_{m=0}^{l-1} R(T^{*m}(y)) = n\}$ and

$$E_{n,l} := \{x \in X^* \cap \mathcal{P}_n(X, T) : \exists y \in X^* \cap \mathcal{P}_l(X^*, T^*) \text{ and } \exists j < R(y)$$

$$\text{such that } \sum_{m=0}^{l-1} R(T^{*m}(y)) = n \text{ and } x = T^j y\}.$$

Then $\mathcal{P}_l(X^*, T^*) = \bigcup_{n \geq l} E_{n,l}^*$ and $\mathcal{P}_n(X, T) \cap X^* = \bigcup_{l \leq n} E_{n,l}$. By Sublemma B we see that for $x \in X^*$,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n} \sum_{x \in \mathcal{P}_n(X, T) \cap X^*} \exp\left[\sum_{h=0}^{n-1} (\phi - s)T^h(x)\right] \\ = \sum_{n=1}^{\infty} \sum_{l \leq n} \sum_{\{x, Tx, \dots, T^{n-1}x\} \subset E_{n,l}} \exp\left[\sum_{h=0}^{n-1} (\phi - s)T^h(x)\right] \\ = \sum_{n=1}^{\infty} \sum_{l \leq n} \sum_{\{y, T^*y, \dots, T^{*l-1}y\} \subset E_{n,l}^*} \exp\left[\sum_{m=0}^{l-1} (\phi^* - sR)T^{*m}(y)\right] \\ = \sum_{l=1}^{\infty} \sum_{\{y, T^*y, \dots, T^{*l-1}y\} \subset \mathcal{P}_l(X^*, T^*)} \exp\left[\sum_{m=0}^{l-1} (\phi^* - sR)T^{*m}(y)\right] = \zeta_{T^*, \phi^* - sR}(1). \end{aligned}$$

We complete the proof. \square

Proof of Lemma 7. (i) By Lemmas 13-14, we have continuity of the function $P_{\text{top}}(T^*, \phi^* - sR)$ on $\text{int}\{s \in \mathbb{R} \mid P_{\text{top}}(T^*, \phi^*) \in \mathbb{R}\}$. Then the existence of a zero $s_0 \geq 0$ of (GBE) follows from the standard argument. Since $P_{\text{top}}(T^*, \phi^* - s_0R) = 0$, by Corollary 1 we have $s_0 \leq P_{\text{top}}(T, \phi)$. If $P_{\text{top}}(T, \phi) < 0$, then we have a contradiction. For (ii), replacing ϕ by $\phi - P_{\text{top}}(T^*, \phi^*)$ allows us to reduce to the case (i). (iii) Since the case $0 \leq P_{\text{top}}(T^*, \phi^*) < \infty$ is covered by Lemma 7(i), we suppose either $P_{\text{top}}(T^*, \phi^*) = \infty$ or $P_{\text{top}}(T^*, \phi^*) < 0$. If $P_{\text{top}}(T^*, \phi^*) = \infty$, then $\sup\{s \in \mathbb{R} : P_{\text{top}}(T^*, \phi^* - sR) = \infty\} = 0$. Hence $\forall s > 0$, $P_{\text{top}}(T^*, \phi^* - sR) < \infty$. Since the function $s \rightarrow P_{\text{top}}(T^*, \phi^* - sR)$ is decreasing and continuous on $\text{int}\{s \in \mathbb{R} \mid P_{\text{top}}(T^*, \phi^* - sR) \in \mathbb{R}\}$, we have $\lim_{s \rightarrow 0} P_{\text{top}}(T^*, \phi^* - sR) = \infty$ so that for sufficiently small $s > 0$, $P_{\text{top}}(T^*, \phi^* - sR) > 0$. On the other hand, it follows from

Corollary 1 that $P_{\text{top}}(T^*, \phi^* - sR) \leq 0$ for $s > P_{\text{top}}(T, \phi)$. Hence we have a zero $s_0 \geq 0$ of (GBE). If $P_{\text{top}}(T^*, \phi^*) < 0$, then $\sup\{s \in \mathbb{R} : P_{\text{top}}(T^*, \phi^* - sR) = \infty\} = P_{\text{top}}(T^*, \phi^*)$. The same argument as those for the previous case allows us to have a zero $s_0 \geq P_{\text{top}}(T^*, \phi^*)$ of (GBE). \square

Proof of Theorem 4. By Lemma 8 we have for $\underline{i} \in I^*$, $\frac{d(\nu v_{\underline{i}})}{d\nu} = \exp[\phi^* - s_0 R]v_{\underline{i}}$. If $\underline{i} = \underline{i}_1 \underline{i}_2 \dots \underline{i}_n$, then $\frac{d(\nu v_{\underline{i}})}{d\nu}(x) = \exp[\sum_{h=0}^{n-1} \phi T^h v_{\underline{i}_1 \underline{i}_2 \dots \underline{i}_n} x - s_0 n]$. Since the property (4) : $R(Tx) = R(x) - 1$ ($R(x) \geq 2$) implies $\underline{i}_2 \underline{i}_3 \dots \underline{i}_n \in I^*$, the equality

$$\frac{d(\nu v_{\underline{i}})}{d\nu} = \frac{d(\nu v_{\underline{i}_1})}{d\nu}(\underline{i}_2 \underline{i}_3 \dots \underline{i}_n x) \frac{d(\nu v_{\underline{i}_2 \underline{i}_3 \dots \underline{i}_n})}{d\nu}(x)$$

allows one to see that $\forall X_i \subset D_1$, $\frac{d(\nu v_i)}{d\nu}(x) = \exp[\phi v_i(x) - s_0](\forall x \in X^*)$. On the other hand, we know that the above equality holds for $X_i \subset B_1$ since $i \in I^*$. Finally, we establish $\forall X_i \in Q$, $\frac{d(\nu T)}{d\nu}|_{X_i}(x) = \exp[s_0 - \phi(x)](\forall x \in X^*)$. It follows from Lemma 2 and Theorem 2.1 in [18] that $s_0 = P_{\text{top}}(T, \phi)$. The assertion $\nu(\bigcup_{i \in I} \partial X_i) = \nu(\bigcup_{i \in I} \partial v_i(X)) = 0$ follows from $\nu(\text{int} X) = 1$, which is obtained by Lemma 8. We complete the proof. \square

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